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A CONTINUUM THEORY FOR ISOTROPIC TWO-PHASE ELASTIC COMPOSITES

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Abstract—The paper presents the effective stiffness theory for isotropic two-phase elastic composites. The theory predicts dispersion of longitudinal and transverse plane time-harmonic travelling waves. The limiting phase velocities at vanishing wave numbers serve in the determination of the elastic moduli of the equivalent homogeneous isotropic medium. These elastic moduli are compared with the effective moduli defined statically.

1. INTRODUCTION

In the conventional method of describing composite media, the composite is replaced by a homogeneous classical continuum. The main problem there consists of finding the effective moduli. As this method of description fails to indicate dispersion of waves, it is not too satisfactory for dynamic problems.

A conceptionally different approach was proposed in [1, 2] for the case of a laminated medium. The respective method, termed the "effective stiffness theory" was later applied to unidirectional fibre-reinforced composites [3–5].

In this paper we shall evolve the effective stiffness theory for a certain type of isotropic two-phase elastic composites. The geometry of the composite is described in Section 2. It is assumed that one phase is formed by inclusions, the other phase by a matrix, and that the composite is macroscopically homogeneous and isotropic. In the interest of simplification the medium is conceived to be a set of identical composite elements. A composite element is formed by a spherical inclusion and an appertaining spherical jacket of the matrix. The interaction between the inclusion and the matrix jacket, as well as the interaction between adjacent composite elements is taken into account by simulating continuity of the displacement vector at the interfaces.† In Section 3, the equations of motion are obtained by the application of Hamilton's principle. Section 4 studies the propagation of plane time-harmonic travelling waves. Both the transverse and the longitudinal waves are dispersive. In Section 5 the limiting phase velocities at vanishing wave numbers are employed in the determination of the elastic moduli of the equivalent simple homogeneous isotropic medium. The elastic moduli thus established are compared with the effective moduli defined statically.

2. KINEMATICS

Consider a composite medium macroscopically homogeneous and isotropic, whose one phase forms more or less spherical inclusions while the other phase forms a matrix (Fig. 1a). Assign to each inclusion a certain neighborhood of the matrix so as to obtain composite elements whose shape comes to resemble spheres. The individual inclusions are generally of different shapes and sizes, and their microscopical array is irregular. The assumption of macroscopical homogeneity and isotropy offers, however, the possibility of approximately replacing the actual composite by a fictitious medium which is formed by identical composite elements consisting of an inclusion surrounded by the jacket of the matrix material (Fig. 1b).‡ If the volume per cent of inclusions is equal to η^3, and r_1 denotes the radius of an average-sized spherical inclusion, the radius of the

†As no continuum without voids or overlapping can be composed of such identical composite elements, continuity of displacements between the neighbouring elements is only in the averaging sense.

‡It is known that if the elastic moduli of the components differ a great deal, there is a considerable scatter in properties of the composite material caused by the different shapes and the lay-out of the inclusions. Here we shall consider sphere-shaped inclusions. Needle-shaped, disk-shaped and spheroid-shaped inclusions will be considered elsewhere.

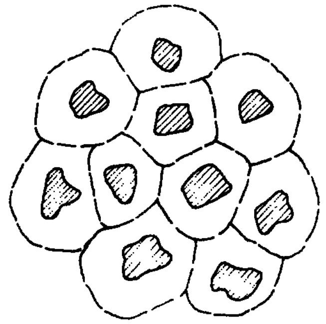

Fig. 1a. Heterogeneous material.

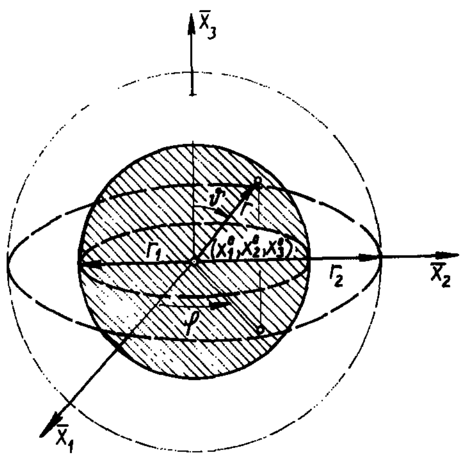

Fig. 1b. Composite element.

composite element is given by

$$r_2 = \frac{r_1}{\eta}.$$

As, however, no continuum without voids or overlapping can in reality be composed of such identical composite spherical elements, the medium will in this sense be a fictitious one.

Let x_i $(i = 1, 2, 3)$ be the global Cartesian coordinates. Consider a composite element (Fig. 1b) with x_{0i} the coordinates of its centre. Introduce in x_{0i} the local Cartesian coordinates $\bar{x}_i$ and the local spherical coordinates r, φ, ϑ, i.e. write the relations

$$\begin{aligned} x_i &= x_{0i} + \bar{x}_i, \quad i = 1, 2, 3 \\ \bar{x}_1 &= r \cos\varphi \sin\vartheta, \\ \bar{x}_2 &= r \sin\varphi \sin\vartheta, \\ \bar{x}_3 &= r \cos\vartheta. \end{aligned} \tag{2.1}$$

In the discussion that follows the Latin indices will take on the values of 1, 2, 3 and summation will be assumed with respect to pairs of identical Latin indices.

We assume the displacement vector $u_i^{(1)}$ in the inclusion to be linearly dependent on $\bar{x}_i$, and the displacement vector $u_i^{(2)}$ in the matrix jacket of the element to be linearly dependent on r, i.e.

$$\begin{aligned} u_i^{(1)}(x_i, t) &= u_{0i}^{(1)}(x_{0j}, t) + \bar{x}_k \psi_{ki}(x_{0j}, t), \\ u_i^{(2)}(x_j, t) &= u_{0i}^{(2)}(x_{0j}, r_2, \varphi, \vartheta, t) + (r - r_2)\omega_i(x_{0j}, \varphi, \vartheta, t). \end{aligned} \tag{2.2}$$

In the above, $u_{0i}^{(1)}(x_{0j}, t)$ denotes the displacement vector at the centre of the composite element, $u_{0i}^{(2)}(x_{0j}, r_2, \varphi, \vartheta, t)$ the displacement vector on the outer surface of the element at a point with the local coordinates r_2, φ, ϑ. $u_{0i}^{(1)}$, $u_{0i}^{(2)}$, ψ_{ki} and ω_i are functions defined only for discrete x_{0i} or r_2, i.e. at the centres or on the surfaces of the composite elements. Since r_1 and r_2 are very small compared with the macroscopical unit length, we shall replace these functions by continuous functions defined for all x_i or $r > 0$. We shall further assume that $u_{0i}^{(1)}$ and $u_{0i}^{(2)}$ can be replaced by a single vector function u_i ($i = 1, 2, 3$) called the gross-displacement so that it will be

$$
\begin{aligned}
u_{0i}^{(1)}(x_j, t) &= u_i(x_j, t), \\
u_{0i}^{(2)}(x_j, r_2, \varphi, \vartheta, t) &= u_i(x_1 + r_2 \cos\varphi \sin\vartheta, x_2 + r_2 \sin\varphi \sin\vartheta, x_3 + r_2 \cos\vartheta, t) \\
&= u_i(x_j, t) + r_2[u_{i,1} \cos\varphi \sin\vartheta + u_{i,2} \sin\varphi \sin\vartheta + u_{i,3} \cos\vartheta].
\end{aligned}
\tag{2.3}
$$

The condition of continuity of displacements on the surface of the inclusion gives for $r = r_1$—with the use of (2.2) to (2.3)—the bond between ω_i and ψ_{ki}, viz.

$$
(r_2 - r_1)\omega_i = \cos\varphi \sin\vartheta(r_2 u_{i,1} - r_1\psi_{1i}) + \sin\varphi \sin\vartheta(r_2 u_{i,2} - r_1\psi_{2i}) + \cos\vartheta(r_2 u_{i,3} - r_1\psi_{3i}). \tag{2.4}
$$

The bond between neighbouring composite elements is guaranteed by the existence of u_i and by relations (2.3). Substitution of (2.4) into (2.2) yields

$$
\begin{aligned}
u_i^{(1)} &= u_i + \bar{x}_j\psi_{ji} \quad \text{for} \quad 0 \leq r \leq r_1, \\
u_i^{(2)} &= u_i + \bar{x}_j H_{ji} \quad \text{for} \quad r_1 \leq r \leq r_2
\end{aligned}
\tag{2.5}
$$

where

$$
H_{ji} = \frac{1}{r}\left[r_2 u_{i,j} + \frac{r - r_2}{r_2 - r_1}(r_2 u_{i,j} - r_1\psi_{ji})\right].
$$

The state of deformation in the medium is now described by the gross-displacement u_i and by the tensor ψ_{ji}.

3. THE EQUATIONS OF MOTION

The strain energy W' of the composite element is defined by

$$
\begin{aligned}
W' &= \iiint_{V^{(1)}} \left[\frac{1}{2}\lambda_1 \epsilon_{ii}^{(1)}\epsilon_{kk}^{(1)} + \mu_1 \epsilon_{ij}^{(1)}\epsilon_{ij}^{(1)}\right] d\bar{x}_1\, d\bar{x}_2\, d\bar{x}_3 \\
&\quad + \iiint_{V^{(2)}} \left[\frac{1}{2}\lambda_2 \epsilon_{ii}^{(2)}\epsilon_{kk}^{(2)} + \mu_2 \epsilon_{ij}^{(2)}\epsilon_{ij}^{(2)}\right] d\bar{x}_1\, d\bar{x}_2\, d\bar{x}_3,
\end{aligned}
\tag{3.1}
$$

$$
\epsilon_{ij}^{(1)} = u_{(i,j)}^{(1)}, \quad \epsilon_{ij}^{(2)} = u_{(i,j)}^{(2)}.
$$

In the above λ_1, μ_1 are the elastic Lamé's constants of the inclusion, and λ_2, μ_2 are the elastic Lamé's constants of the matrix. $V^{(1)}$ and $V^{(2)}$ are, respectively, the volume of the inclusion ($0 \leq r \leq r_1$) and the volume of the matrix jacket ($r_1 \leq r \leq r_2$). The differentiation in (3.1) is understood to be with respect to $\bar{x}_i$. Using (2.5) we obtain after fairly lengthy calculations the strain energy density

$$
W = \frac{1}{\frac{4}{3}\pi r_2^3} W'
$$

in the form

$$
W = \frac{1}{2}A_{ijkl}\epsilon_{ij}\epsilon_{kl} + \frac{1}{2}B_{ijkl}\epsilon_{ij}\gamma_{kl} + \frac{1}{2}C_{ijkl}\gamma_{ij}\gamma_{kl} \tag{3.2}
$$

where

$$
\epsilon_{ij} = u_{(i,j)}, \qquad \gamma_{ij} = u_{j,i} - \psi_{ij}.
$$

The constant tensors A_{ijkl}, B_{ijkl}, C_{ijkl} have the following non-zero components:

$$
\begin{aligned}
&A_{1111} = A_{2222} = A_{3333} = \eta^3(\lambda_1 + 2\mu_1) + (1-\eta^3)(\lambda_2 + 2\mu_2),\\
&A_{1122} = A_{2211} = A_{1133} = A_{3311} = A_{2233} = A_{3322} = \eta^3\lambda_1 + (1-\eta^3)\lambda_2,\\
&A_{1212} = A_{2112} = A_{2121} = A_{1221} = A_{1313} = A_{3113} = A_{3131} = A_{1331}\\
&\qquad = A_{2323} = A_{3223} = A_{3232} = A_{2332} = \eta^3\mu_1 + (1-\eta^3)\mu_2,\\
&B_{1111} = B_{2222} = B_{3333} = 2\eta^3[(\lambda_2 + 2\mu_2) - (\lambda_1 + 2\mu_1)],\\
&B_{1122} = B_{2211} = B_{1133} = B_{3311} = B_{2233} = B_{3322} = 2\eta^3(\lambda_2 - \lambda_1),\\
&B_{1212} = B_{2112} = B_{2121} = B_{1221} = B_{1313} = B_{3113} = B_{3131} = B_{1331}\\
&\qquad = B_{2323} = B_{3223} = B_{3232} = B_{2332} = 2\eta^3(\mu_2 - \mu_1),\\
&C_{1111} = C_{2222} = C_{3333} = \eta^3(\lambda_1 + 2\mu_1) + (3V - \eta^3)\lambda_2 + (8V - 2\eta^3)\mu_2,\\
&C_{1122} = C_{2211} = C_{1133} = C_{3311} = C_{2233} = C_{3322} = \eta^3\lambda_1 + (V - \eta^3)\lambda_2 + V\mu_2,\\
&C_{1212} = C_{2121} = C_{1313} = C_{3131} = C_{2323} = C_{3232} = \eta^3\mu_1 + V\lambda_2 + (6V - \eta^3)\mu_2,\\
&C_{1221} = C_{2112} = C_{1331} = C_{3113} = C_{2332} = C_{3223} = \eta^3\mu_1 + V\lambda_2 + (V - \eta^3)\mu_2,
\end{aligned}
\tag{3.3}
$$

where

$$\eta = \frac{r_1}{r_2}, \qquad V = \frac{\eta^2}{5(1-\eta)}.$$

The kinetic energy of the composite element is defined by

$$K' = \frac{1}{2}\sum_{i=1}^{3}\left[\iiint_{V^{(1)}} \rho_1 \dot{u}_i^{(1)2}\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3 + \iiint_{V^{(2)}} \rho_2 \dot{u}_i^{(2)2}\,\mathrm{d}\bar{x}_1\,\mathrm{d}\bar{x}_2\,\mathrm{d}\bar{x}_3\right]. \tag{3.4}$$

ρ_1 and ρ_2 are, respectively, the density of the inclusion and the density of the matrix. The dot above a quantity denotes the derivative with respect to time t. Using (2.5) we obtain the kinetic energy of unit volume of the composite

$$K = \frac{1}{\frac{4}{3}\pi r_2^3}K'$$

in the form

$$
\begin{aligned}
K = \frac{1}{2}\sum_{i=1}^{3}\Big\{&\bar{\rho}\dot{u}_i^2 + J(\dot{\psi}_{1i}^2 + \dot{\psi}_{2i}^2 + \dot{\psi}_{3i}^2) + \rho_2\Big[\frac{X}{2}(\dot{u}_{i,1}^2 + \dot{u}_{i,2}^2 + \dot{u}_{i,3}^2)\\
&+ Z(\dot{u}_{i,1}\dot{\psi}_{1i} + \dot{u}_{i,2}\dot{\psi}_{2i} + \dot{u}_{i,3}\dot{\psi}_{3i})\Big]\Big\},
\end{aligned}
\tag{3.5}
$$

where

$$
\begin{aligned}
&\bar{\rho} = \eta^3\rho_1 + (1-\eta^3)\rho_2, \qquad J = \frac{r_1^2}{5}\left[\eta^3\rho_1 + \frac{\rho_2}{6}(1 + 2\eta + 3\eta^2 - 6\eta^3)\right],\\
&X = \frac{r_1^2}{15\eta^2}(6 - 3\eta - 2\eta^2 - \eta^3), \qquad Z = \frac{r_1^2}{30\eta}(3 - 5\eta + 5\eta^4 - 3\eta^5).
\end{aligned}
\tag{3.6}
$$

Let V denote a fixed regular region, and t_1, t_2 fixed times. For independent variations of u_i, ψ_{ij} for which

$$\delta u_i = \delta\psi_{ij} = 0$$

on the surface S of region V, Hamilton's principle is of the form

$$\delta\int_{t_1}^{t_2}\int_V (K - W)\,\mathrm{d}V\,\mathrm{d}t = 0. \tag{3.7}$$

The sought equations of motion are Euler's conditions of the variational principle (3.7). After calculations using (3.2), (3.5) we obtain a total of twelve equations of motion, viz.

$$\begin{aligned} &a_1 u_{1,11} + a_2(u_{1,22} + u_{1,33}) + a_3(u_{2,21} + u_{3,31}) + a_4\psi_{11,1} + a_6(\psi_{22,1} + \psi_{33,1}) \\ &\quad + a_6(\psi_{12,2} + \psi_{13,3}) + a_7(\psi_{21,2} + \psi_{31,3}) + a_8\ddot{u}_1 + a_9(\ddot{u}_{1,11} + \ddot{u}_{1,22} + \ddot{u}_{1,33}) \\ &\quad + a_{10}(\ddot{\psi}_{11,1} + \ddot{\psi}_{21,2} + \ddot{\psi}_{31,3}) = 0, \end{aligned} \tag{3.8}$$

$$a_4 u_{1,1} + a_6(u_{2,2} + u_{3,3}) + a_{11}\psi_{11} + a_{12}(\psi_{22} + \psi_{33}) + a_{13}\ddot{\psi}_{11} + a_{10}\ddot{u}_{1,1} = 0, \tag{3.9}$$

$$a_6 u_{1,2} + a_7 u_{2,1} + a_{14}\psi_{12} + a_{15}\psi_{21} + a_{13}\ddot{\psi}_{12} + a_{10}\ddot{u}_{2,1} = 0, \tag{3.10}$$

$$a_6 u_{2,1} + a_7 u_{1,2} + a_{14}\psi_{21} + a_{15}\psi_{12} + a_{13}\ddot{\psi}_{21} + a_{10}\ddot{u}_{1,2} = 0. \tag{3.11}$$

The remaining eight equations are obtained from (3.8) to (3.11) by cyclic permutation of the indices 1, 2, 3. We have introduced the following notation in (3.8) to (3.11):

$$\begin{aligned} &a_1 = (3V+1)\lambda_2 + (8V+2)\mu_2, && a_2 = V\lambda_2 + (6V+1)\mu_2, \\ &a_3 = (2V+1)(\lambda_2 + \mu_2), && a_4 = -V(3\lambda_2 + 8\mu_2), \\ &a_6 = -V(\lambda_2 + \mu_2), && \\ &a_7 = -V(\lambda_2 + 6\mu_2), && a_8 = -\bar{\rho}, \\ &a_9 = \frac{X}{2}\rho_2, && a_{10} = \frac{Z}{2}\rho_2, \\ &a_{11} = \eta^3(\lambda_1 + 2\mu_1) + (3V - \eta^3)\lambda_2 + (8V - 2\eta^3)\mu_2, && \\ &a_{12} = \eta^3\lambda_1 + (V - \eta^3)\lambda_2 + V\mu_2, && a_{13} = J, \\ &a_{14} = \eta^3\mu_1 + V\lambda_2 + (6V - \eta^3)\mu_2, && \\ &a_{15} = \eta^3\mu_1 + V\lambda_2 + (V - \eta^3)\mu_2. && \end{aligned} \tag{3.12}$$

If we take the limit $\eta \to 0$, $r_2 \to 0$ in the equations of motion, we arrive at classical Lamé's equations of motion of a simple elastic medium with Lamé's constants λ_2, μ_2.

4. PROPAGATION OF PLANE HARMONIC WAVES

The equations of motion obtained above will be used to study the propagation of plane time-harmonic travelling waves. We assume the solution to the equations to be of the form

$$u_i = U_i\, e^{ik(x_1 - ct)}, \qquad \psi_{ij} = \Psi_{ij}\, e^{ik(x_1 - ct)}.$$

Here U_i, Ψ_{ij} are the constant amplitudes, k is the wave number, c the phase velocity. After substitution the set of twelve equations of motion will decompose into four systems: the first system will contain U_2, Ψ_{12}, Ψ_{21}, the second U_3, Ψ_{13}, Ψ_{31}, the third U_1, Ψ_{11}, Ψ_{22}, Ψ_{33} and the fourth only Ψ_{23}, Ψ_{32}. The first two systems of equations describe the transverse waves, the third represents the longitudinal wave and the fourth system describes the twisting micro-wave. The condition of non-zero amplitudes is that the determinants of these systems of equations should be zero. At the same time, those are the conditions affording a relation between c and k, i.e. they are the sought dispersion relations. For transverse waves the dispersion relation turns out to be

$$\begin{vmatrix} a_2 + c^2 a_8 - c^2k^2 a_9, & a_6, & a_7 - c^2k^2 a_{10} \\ a_6, & a_{14} - c^2k^2 a_{13}, & a_{15} \\ a_7 - c^2k^2 a_{10}, & a_{15}, & a_{14} - c^2k^2 a_{13} \end{vmatrix} = 0. \tag{4.1a}$$

The dispersion relation for the longitudinal wave is

$$\begin{vmatrix} a_1 + c^2 a_8 - c^2k^2 a_9, & a_4 - c^2k^2 a_{10}, & 2a_6 \\ a_4 - c^2k^2 a_{10}, & a_{11} - c^2k^2 a_{13}, & 2a_{12} \\ a_6, & a_{12}, & (a_{11} + a_{12}) - c^2k^2 a_{13} \end{vmatrix} = 0. \tag{4.1b}$$

The dispersion of the twisting wave is given by

$$\begin{vmatrix} a_{14}-c^2k^2a_{13}, & a_{15} \\ a_{15}, & a_{14}-c^2k^2a_{13} \end{vmatrix} = 0. \tag{4.1c}$$

We shall evaluate the limit phase velocity ${}^0c$ for $k \to 0$. From (4.1a) we obtain

$${}^0c^2 = \frac{1}{a_8}\left\{-a_2 + \frac{2a_6a_7a_{15} - a_{14}(a_6^2 + a_7^2)}{a_{15}^2 - a_{14}^2}\right\}. \tag{4.2}$$

(4.1b) gives

$${}^0c^2 = \frac{1}{a_8}\left\{-a_1 + \frac{a_4^2(a_{11}+a_{12}) + 2a_6^2a_{11} - 4a_6a_4a_{12}}{a_{11}^2 + a_{11}a_{12} - 2a_{12}^2}\right\}. \tag{4.3}$$

From (4.1c) it follows $c \to \infty$ for $k \to 0$.

The lowest modes of the dispersion curves for the transverse wave are shown in Fig. 2, and for the longitudinal wave in Fig. 3. In place of c, k there are plotted the dimensionless quantities β, ξ

$$\beta = c\left(\frac{\rho_2}{\mu_2}\right)^{1/2}, \qquad \xi = kr_1.$$

We also write that

$$\gamma = \frac{\mu_1}{\mu_2}, \qquad \vartheta = \frac{\rho_1}{\rho_2}, \qquad \nu_\alpha = \frac{\lambda_\alpha}{2(\lambda_\alpha + \mu_\alpha)}, \qquad \alpha = 1, 2.$$

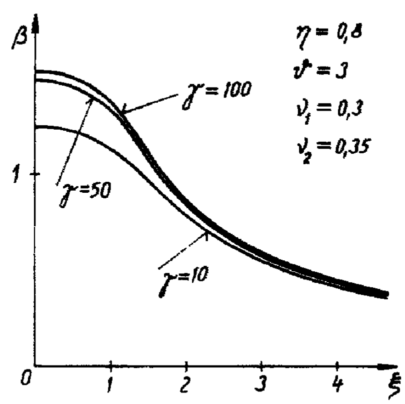

Fig. 2. Dispersion of transverse waves.

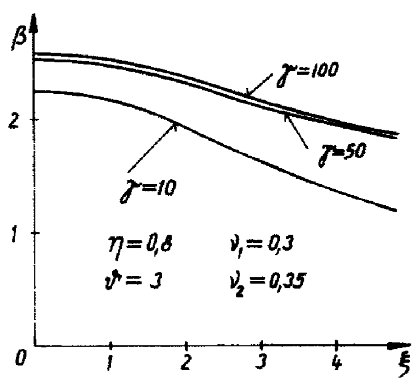

Fig. 3. Dispersion of longitudinal waves.

The curves in Figs. 2 and 3 are drawn for the values:

$$\eta = 0{\cdot}8; \quad \vartheta = 3; \quad \nu_1 = 0{\cdot}3; \quad \nu_2 = 0{\cdot}35;$$

$$\gamma = 10; \quad \gamma = 50; \quad \gamma = 100.$$

For this case the volume per cent of inclusions is $\eta^3 = 0{\cdot}512$. It is seen that the transverse waves display a stronger dispersion than the longitudinal waves.

5. EFFECTIVE MODULI

In the preceding section we have deduced the dispersion relations for plane waves in the effective stiffness model. The approximate dispersion curves obtained in the manner described cannot, however, be compared with exact curves, for no exact elasticity solutions exist.

We can, however, compare the moduli $\bar{\lambda}$, $\bar{\mu}$ of the equivalent isotropic homogeneous simple medium in which the transverse wave propagates with a constant phase velocity ${}^0c$ defined by expression (4.2) and the longitudinal wave with a constant phase velocity ${}^0c$ defined by expression (4.3), with the effective moduli defined statically. If the density of the homogeneous isotropic medium is equal to $\bar{\rho}$ from (3.6), we have from (4.2), (4.3) that

$$\begin{aligned} \bar{\mu} &= a_2 + \frac{2a_6a_7a_{15} - a_{14}(a_6^2 + a_7^2)}{a_{14}^2 - a_{15}^2}, \\ \bar{\lambda} + 2\bar{\mu} &= a_1 - \frac{a_4^2(a_{11} + a_{12}) + 2a_6^2a_{11} - 4a_4a_6a_{12}}{a_{11}^2 + a_{11}a_{12} - 2a_{12}^2}. \end{aligned} \tag{5.1}$$

Note that $\bar{\lambda}$ and $\bar{\mu}$ defined in (5.1) depend on λ_1, μ_1, λ_2, μ_2 and η. They depend on r_1 only through the intermediary of η, i.e. $\bar{\lambda}$ and $\bar{\mu}$ do not change with changing r_1 provided the ratio r_1/r_2 continues constant. Hence (5.1), (4.2), (4.3) continue to apply even in the case of different inclusion sizes. If r_2 can be arbitrarily small at constant r_1/r_2, it is already possible to construct from variously large composite spherical elements a continuum without voids or overlapping which is macroscopically homogeneous and isotropic. For such a model, Z. Hashin [6] established the exact effective volume modulus $\bar{\kappa}_e$ (cf. eqn (38) in [6]) and the exact bounds for the effective shear modulus. As the calculation of these bounds is fairly complicated, Z. Hashin presents—for the case that the phase moduli do not differ too much from one another—an approximate value of the effective shear modulus $\bar{\mu}_a$ (eqn (54) in [6]) which always lies between the exact bounds. In [7] Z. Hashin and S. Shtrikman derived the upper and the lower bounds for the effective shear modulus, μ_l and μ_u of a two-phase composite with an arbitrary geometry, provided that the composite is macroscopically homogeneous and isotropic.

We shall carry out a comparison for the tungsten-carbide-cobalt alloy considered in [6]. The alloy consists of tungsten-carbide particles embedded in a matrix of cobalt. The moduli of the particles are $\mu_1 = 41{\cdot}8 \times 10^6$ psi, $\kappa_1 = 60{\cdot}7 \times 10^6$ psi, and the moduli of the matrix are $\mu_2 = 11{\cdot}5 \times 10^6$ psi, $\kappa_2 = 25{\cdot}0 \times 10^6$ psi. The volume modulus of the composite $\bar{\kappa}$,

$$\bar{\kappa} = \bar{\lambda} + \frac{2}{3}\bar{\mu},$$

calculated from (5.1), differs for $\eta^3 \in (0, 1)$ from the exact value $\bar{\kappa}_e$ obtained from (38), [6] by 1 per cent maximum. $\bar{\mu}$ calculated from (5.1) for $\eta^3 \in (0, 1)$ lies within Hashin–Shtrikman bounds μ_l, $\mu_u = \mu_a$.

For the alloy considered, $\gamma = 3{\cdot}65$. Consider a composite material with $\gamma = 100$, $\nu_1 = \nu_2 = 0{\cdot}3$. Again, the deviation of $\bar{\kappa}$ from $\bar{\kappa}_e$ is less than 1 per cent. Now Hashin–Shtrikman bounds for the shear modulus are very broad and $\bar{\mu}$ lies within these bounds being very near the lower bound $\bar{\mu}_l = \bar{\mu}_a$.

We may therefore claim that the moduli $\bar{\kappa}$, $\bar{\mu}$ defined in (5.1) afford approximate values of the effective moduli defined statically. By substituting (3.12) into (5.1) we obtain for $\bar{\mu}$ the simple expression

$$\bar{\mu} = \left\{1 + \frac{V\eta^3(\gamma - 1)(2\delta_2 + 3)}{2\eta^3(\gamma - 1) + V(2\delta_2 + 3)}\right\}\mu_2$$

where

$$\delta_2 = \frac{2(1 - \nu_2)}{1 - 2\nu_2}, \qquad \nu_2 = \frac{\lambda_2}{2(\lambda_2 + \mu_2)}.$$

The expression of $\bar{\kappa}$ would turn out to be more complicated and we shall not present it here. As a matter of fact, the exact value of $\bar{\kappa}_e$ is available in (38), [6].

6. CONCLUSION

The paper evolves the effective stiffness theory for an isotropic two-phase elastic composite material. One phase of this material is formed by spherical elastic inclusions, the other phase by an elastic matrix. The theory predicts the dispersion of plane time-harmonic travelling waves. The phase velocities for long wave lengths define the elastic moduli which are found to be very close to the effective moduli defined statically.

The effective stiffness method involves a homogeneous higher-order continuum with microstructure. It bears a close resemblance to some elasticity theories with microstructure. In the model presented here, we have two tensors of deformation: ϵ_{ij} and γ_{ij}. In Refs. [1–3, 5] there are four deformation tensors: ϵ_{ij}, γ_{ij},

$$\vartheta_{ijk} = \gamma_{ij,k} \quad \text{and} \quad \kappa_{ijk} = \psi_{ij,k}.$$

In Mindlin's microstructure elasticity theory [9] three tensors of deformation were introduced: ϵ_{ij}, γ_{ij} and κ_{ijk}. A detailed comparison would reveal that the model presented in this paper is a special kind of Mindlin's theory. The effective stiffness model yields, however, the elastic material tensors as functions of the elastic moduli of constituents and of the geometrical lay-out of inclusions, while in Mindlin's theory these tensors remain undetermined.

In Refs. [1–5] and here Hamilton's principle was used. Starting from other dynamical variational principles formulated through convolutions we could proceed in the same way for thermoelastic [8] and viscoelastic composite materials.

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